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Mixed integrable $SU(N)$ vertex model with arbitrary twists

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Abstract

We consider the quantum inverse scattering method for several mixed integrable models based on the rational $SU(N)$ R-matrix with general toroidal boundary conditions. This includes systems whose Hilbert spaces are invariant by the discrete representations of the group $SU(2)$ and the non-compact group $SU(1,1)$ as well as the conjugate representation of the $SU(N)$ symmetry. Introducing certain transformations on the quantum spaces we are able to solve generalized impurity problems including those related to singular matrices.

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1 Introduction

The quantum version of the inverse scattering method is nowadays accepted as the foundation of the algebraic theory of integrable models of quantum field theory and classical statistical mechanics in $(1+1)$ dimensions [1, 2]. This approach has made possible not only the discovery of new important models such as the Heisenberg chain with arbitrary spin [3, 4] but also precipitated the notion of quantum group symmetry [5, 6].

An essential ingredient of the quantum inverse scattering method is frequently denominated the Yang-Baxter algebra whose generators can be thought as the elements of square $N \times N$ matrix $\mathcal{T}_{\mathcal{A}}(\lambda)$. The symbol $\mathcal{A}$ emphasizes such N-dimensional auxiliary space and λ denotes the spectral parameter. This associative algebra is then generated by the following quadratic relations

$$\check{R}(\lambda-\mu)\mathcal{T}_{\mathcal{A}}(\lambda)\otimes\mathcal{T}_{\mathcal{A}}(\mu)=\mathcal{T}_{\mathcal{A}}(\mu)\otimes\mathcal{T}_{\mathcal{A}}(\lambda)\check{R}(\lambda-\mu), \tag{1}$$

where $\check{R}(\lambda)$ is a $N^2 \times N^2$ matrix whose elements are complex numbers. Here we are assuming that this R-matrix is additive with respect the spectral parameters.

The R-matrix $\check{R}(\lambda)$ is required to satisfy a sufficient condition, that guarantees the associativity of the algebra (1), known as the Yang-Baxter equation

$$\check{R}_{12}(\lambda)\check{R}_{23}(\lambda+\mu)\check{R}_{12}(\mu)=\check{R}_{23}(\mu)\check{R}_{12}(\lambda+\mu)\check{R}_{23}(\lambda), \tag{2}$$

where $\check{R}_{ab}(\lambda)$ denotes the action of the R-matrix on the tensor product space $\mathcal{A}_a \otimes \mathcal{A}_b$.

It is not difficult to see that the trace of $\mathcal{T}_{\mathcal{A}}(\lambda)$ over the auxiliary space

$$T(\lambda)=\mathrm{Tr}_{\mathcal{A}}\left[\mathcal{T}_{\mathcal{A}}(\lambda)\right], \tag{3}$$

gives origin to a commutative family of operators, i.e. $[T(\lambda),T(\mu)]=0$ for arbitrary values of λ and μ. In this sense, $T(\lambda)$ can be regarded as the generating function of quantum integrals of motion.

One possible representation of $\mathcal{T}_{\mathcal{A}}(\lambda)$ is provided by numeric $N\times N$ matrix $\mathcal{G}_{\mathcal{A}}$ whose entries do not depend on λ and that satisfies the relation

$$\left[\check{R}(\lambda),\mathcal{G}_{\mathcal{A}}\otimes\mathcal{G}_{\mathcal{A}}\right]=0, \tag{4}$$

and therefore being direct related to the underlying symmetries of $\check{R}(\lambda)$.

General representations of (1) for a given R-matrix depend on the spectral parameter λ and are often called $\mathcal{L}$-operators $\mathcal{L}_{\mathcal{A}j}(\lambda)$. They are viewed as matrices on the auxiliary space $\mathcal{A}$ whose elements are operators on another space V_j known as quantum space. The simplest one occurs when the auxiliary space $\mathcal{A}$ and the quantum space V_j are isomorphic, since in this case Eq.(2) becomes equivalent to Eq.(1) provided we set

$$\mathcal{L}_{\mathcal{A}j}(\lambda) = P_{\mathcal{A}j}\check{R}(\lambda), \tag{5}$$

where $P_{\mathcal{A}j}$ is the exchange operator on the space $\mathcal{A} \otimes V_j$.

An extremely important property of the Yang-Baxter algebra is that the tensor product of two representations is still a representation of this algebra. For example, let $\mathcal{L}_{\mathcal{A}j} \in \prod_{j=1}^{L} \otimes V_j$ for L distinct $\mathcal{L}$-operators which satisfy the Yang-Baxter algebra with the same R-matrix. Then the following ordered tensor product of $\mathcal{L}$-operators, denominated monodromy matrix [1, 2]

$$\mathcal{T}_{\mathcal{A}}(\lambda) = \mathcal{G}_{\mathcal{A}}\mathcal{L}_{\mathcal{A}L}(\lambda)\mathcal{L}_{\mathcal{A}L-1}(\lambda)\dots\mathcal{L}_{\mathcal{A}1}(\lambda), \tag{6}$$

is also a representation of the same Yang-Baxter algebra.

In the context of classical vertex models of statistical mechanics $\mathcal{L}_{\mathcal{A}j}(\lambda)$ represents the possible Boltzmann weights at the local site j of a square lattice of size L, $\mathcal{G}_{\mathcal{A}}$ play the role of possible toroidal boundary conditions compatible with integrability [7, 8] and the operator $T(\lambda)$ turns out to be the corresponding row-to-row transfer matrix. The purpose of this paper is to investigate the eigenvalue problem for the transfer matrix $T(\lambda)$ of such mixed vertex models whose underlying R-matrix is the simplest rational solution of the Yang-Baxter equation invariant by the fundamental representation of $SU(N)$, namely [9, 10]

$$\check{R}_{ab}(\lambda) = \eta \sum_{\alpha=1}^{N} \hat{e}_{\alpha\alpha}^{(a)} \otimes \hat{e}_{\alpha\alpha}^{(b)} + \lambda \sum_{\alpha,\beta=1}^{N} \hat{e}_{\alpha\beta}^{(a)} \otimes \hat{e}_{\beta\alpha}^{(b)}, \tag{7}$$

where $\hat{e}_{\alpha\beta}^{(a)}$ are the $N \times N$ Weyl matrices acting on the space $\mathcal{A}_a$ and η is the so-called quasi-classical parameter. It turns out that the admissible boundary representations $\mathcal{G}_{\mathcal{A}}$ for this

R-matrix are arbitrary $N \times N$ matrices that will be represented by

$$\mathcal{G}_{\mathcal{A}} = \sum_{\alpha,\beta=1}^{N} g_{\alpha\beta} \hat{e}_{\alpha\beta}^{(\mathcal{A})}. \tag{8}$$

The purpose of this paper is twofold. On the one hand, we extend our recent efforts [11] in solving fundamental $SU(N)$ vertex models with rather general non-diagonal toroidal boundary conditions to other interesting group invariance of the quantum space that are not isomorphic to the fundamental representation of the $SU(N)$ symmetry. Secondly, we explore the consequences of the existence of transformations on the quantum space to clarify possible relations between the eigenvectors of $T(\lambda)$ with diagonal and non-diagonal boundary conditions. This fact makes possible the solution of generalized mixed vertex models that combine both boundary and spectral dependent representations at any site of the chain.

This work has been organized as follows. In section 2 we present the algebraic Bethe ansatz solution of $SU(2)$ mixed vertex models when their Hilbert spaces are invariant by higher spin representations of $SU(2)$ and the discrete $D^{+}(k)$ representation of the non-compact group $SU(1,1)$. In section 3 we discuss similar analysis for the isotropic vertex model that mixes the fundamental and the conjugate representations of $SU(N)$. In section 4 we show that suitable transformations on the quantum space are essential for the solution of general classes of mixed vertex models. This includes those whose boundary representations are singular as well as the integrable spin-S Heisenberg model with the most arbitrary non-diagonal boundary conditions. The appendix A is reserved for the study of the completeness of the Hilbert space of some of the mixed vertex models mentioned above with $L = 2$. In appendix B we present the explicit expressions of the quantum transformations for all the mixed systems described in this paper.

2 Mixed $SU(2)$ vertex models

The purpose of this section is to solve the eigenvalues problem for the transfer matrix (3) of mixed vertex models whose underlying R-matrix (7) is $SU(2)$ invariant. One natural way of producing such mixed vertex model is by choosing $\mathcal{L}$-operators intertwining between general

representations of $SU(2)$. The other manner is to look for realizations of the Yang-Baxter algebra (1) in terms of different algebraic structures for the quantum space V_j, even those based on non-compact groups. In what follows we shall explore these both possibilities.

2.1 The higher spin realization

The approach of using arbitrary representations of $SU(2)$ to built up mixed vertex models goes probably back to the work by Andrei and Johannesson [12] who studied the spin-1/2 Heisenberg model in the presence of an impurity of spin-S. Latter on, de Vega and Woyanorovich [13] have used similar idea to construct integrable Heisenberg chains with alternating spins resembling ferrimagnetic models. However, those works have assumed explicitly periodic boundary conditions and therefore not dealt with the most general forms of integrable impurities that are going to be considered here.

The higher spin realization of (1) is built from the local $\mathcal{L}$-operators [3, 4]

$$\mathcal{L}_{\mathcal{A}j}^{(\frac{1}{2},S)}(\lambda) = \begin{pmatrix} \left[\lambda + \frac{\eta}{2}\right] + \eta S_j^z & \eta S_j^- \\ \eta S_j^+ & \left[\lambda + \frac{\eta}{2}\right] - \eta S_j^z \end{pmatrix}, \tag{9}$$

which act in the tensor product of the auxiliary space of spin-1/2 and the space V_j carrying the spin-S representation at the j-th site. The construction of the transfer matrix is standard as described in the introduction and it is given by

$$T_{1/2}^{(S_1,\ldots,S_L)}(\lambda) = \mathrm{Tr}_{\mathcal{A}}\left[\mathcal{G}_{\mathcal{A}}\mathcal{L}_{\mathcal{A}L}^{(\frac{1}{2},S_L)}(\lambda)\ldots\mathcal{L}_{\mathcal{A}1}^{(\frac{1}{2},S_1)}(\lambda)\right]. \tag{10}$$

In order to diagonalize this operator by the quantum inverse scattering method it is desirable the existence of local vacuum vectors $|0\rangle_j$ such that the action of the monodromy matrix $\mathcal{T}_{\mathcal{A}}(\lambda)$ on the global state

$$|0\rangle = \prod_{j=1}^{L} \otimes |0\rangle_j\,, \tag{11}$$

gives as result a triangular matrix.

Since the boundary matrix $\mathcal{G}_{\mathcal{A}}$ is arbitrary, the tensor product of the standard highest spin-S_j states does not work as an appropriate global reference state. Following our previous work

[11], suitable local states can be found by introducing a set of Baxter's gauge transformations M_j [14] on the spin-1/2 auxiliary space such that

$$\widetilde{\mathcal{L}}_{\mathcal{A}j}^{(\frac{1}{2},S_j)}(\lambda) = M_{j+1}^{-1}\mathcal{L}_{\mathcal{A}j}^{(\frac{1}{2},S_j)}(\lambda)M_j, \tag{12}$$

where M_j is an invertible 2×2 matrix whose entries are denoted by

$$M_j = \begin{pmatrix} x_j & r_j \\ y_j & s_j \end{pmatrix}. \tag{13}$$

In terms of these new $\widetilde{\mathcal{L}}$-operators transfer matrix (10) becomes

$$T_{1/2}^{(S_1,\dots,S_L)}(\lambda) = \mathrm{Tr}_{\mathcal{A}}\left[M_1^{-1}\mathcal{G}_{\mathcal{A}}M_{L+1}\widetilde{\mathcal{T}}_{\mathcal{A}}(\lambda)\right], \tag{14}$$

where the $\widetilde{\mathcal{T}}_{\mathcal{A}}(\lambda)$ is the transformed monodromy matrix $\widetilde{\mathcal{T}}_{\mathcal{A}}(\lambda) = \widetilde{\mathcal{L}}_{\mathcal{A}L}^{(\frac{1}{2},S_L)}(\lambda)\dots\widetilde{\mathcal{L}}_{\mathcal{A}1}^{(\frac{1}{2},S_1)}(\lambda)$.

We now search for gauge transformations M_j in such way that $\widetilde{\mathcal{L}}_{\mathcal{A}j}^{(\frac{1}{2},S_j)}(\lambda)$ is annihilated by say its lower left element for general values of the spectral parameter λ. This property turns out to be equivalent to the following condition for the ratio $p_j = \frac{x_j}{y_j}$

$$\left\{\left[\lambda+\frac{\eta}{2}\right](p_{j+1}-p_j) - \eta S_j^z(p_{j+1}+p_j) + \eta S_j^+ p_{j+1}p_j - \eta S_j^-\right\}|0\rangle_j^{(S_j)} = 0. \tag{15}$$

This problem can be solved by expanding the vector $|0\rangle_j^{(S_j)}$ in terms of simultaneous eigenkets $|S_j, m_j\rangle$ of $\vec{S}_j^2$ and S_j^z,

$$|0\rangle_j^{(S_j)} = \sum_{m_j=-S_j}^{S_j} C(S_j,m_j)\,|S_j,m_j\rangle. \tag{16}$$

Substituting this ansatz in Eq.(15) and by taking into account the well known action of the angular momenta ladder operators $S_j^{\pm}$ on the basis $|S_j, m_j\rangle$ we find $(2S+1)$ restrictions for the coefficients $C(S_j, m_j)$ given by

$$\sum_{m_j=-S_j}^{S_j}[-(p_{j+1}+p_j)m_jC(S_j,m_j) + p_jp_{j+1}C(S_j,m_j-1)\sqrt{(S_j+m_j)(S_j-m_j+1)}$$
$$(\lambda+\frac{\eta}{2})(p_{j+1}-p_j)C(S_j,m_j) - C(S_j,m_j+1)\sqrt{(S_j-m_j)(S_j+m_j+1)}]\,|S_j,m_j\rangle = 0. \tag{17}$$

A necessary condition for the solution of these equations such that $|0\rangle_j^{(S_j)}$ is independent of the spectral parameter occurs when $p_{j+1} = p_j$ and therefore the ratio $\frac{x_j}{y_j}$ for $j = 1, \dots, L$ is keep fixed. If we substitute this condition in Eq.(17) we end up with recurrence relations for the coefficients $C(S_j, m_j)$ which can be solved in terms of an arbitrary normalization $C(S_j, S_j)$ constant. The final result for the local vacuum $|0\rangle_j^{(S_j)}$ can then be written as follows

$$|0\rangle_j^{(S_j)} = \sum_{m_j=-S_j}^{S_j} \sqrt{\frac{(2S_j)!}{(S_j+m_j)!(S_j-m_j)!}} p_j^{m_j - S_j} C(S_j, S_j) \left|S_j, m_j\right\rangle, \tag{18}$$

and the action of the operator $\widetilde{\mathcal{L}}_{\mathcal{A}j}^{(S_j)}(\lambda)$ in this state is given by

$$\widetilde{\mathcal{L}}_{\mathcal{A}j}^{(S_j)}(\lambda) \, |0\rangle_j^{(S_j)} = \begin{pmatrix} \left(\lambda + \frac{\eta}{2} + \eta S_j\right) \frac{y_j}{y_{j+1}} \, |0\rangle_j^{(S_j)} & \# \\ 0 & \left(\lambda + \frac{\eta}{2} - \eta S_j\right) \frac{y_{j+1}}{y_j} \frac{\det(M_j)}{\det(M_{j+1})} \, |0\rangle_j^{(S_j)} \end{pmatrix}, \tag{19}$$

where the symbol # denotes non-null states.

The constant ratio p_j is then selected out by imposing that the transformed boundary $M_1^{-1} \mathcal{G}_{\mathcal{A}} M_{L+1}$ matrix becomes a diagonal matrix. This restriction lead us to two possible values for $p_j = p^{(\pm)}$, namely

$$p^{(\pm)} = \frac{(g_{11} - g_{22}) \pm \sqrt{(g_{11} - g_{22})^2 + 4 g_{12} g_{21}}}{2 g_{21}}, \tag{20}$$

and consequently with the help of Eqs.(11,18) to two choices for the global reference state $|0\rangle^{(\pm)}$.

We now have the basic ingredients to turn to the algebraic Bethe ansatz solution of the eigenvalue problem associated to $T_{1/2}^{(S_1, \dots, S_L)}(\lambda)$. The first step is to seek for a convenient representation for the monodromy matrix $\widetilde{\mathcal{T}_{\mathcal{A}}}(\lambda)$. Previous experience in dealing with the quantum inverse scattering method for two-dimensional auxiliary spaces [1, 2] suggest us to adopt the form

$$\widetilde{\mathcal{T}_{\mathcal{A}}}(\lambda) = \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{pmatrix}. \tag{21}$$

The property (19) help us to identify the elements of the transformed monodromy matrix that acts as particle creation and annihilation fields over the pseudo-vacuum $|0\rangle^{(\pm)}$. We see

that $\widetilde{B}(\lambda)$ play the role of creation field and $\widetilde{C}(\lambda)$ is an annihilation operator thanks to the property $\widetilde{C}(\lambda)\left|0\right\rangle^{(\pm)} = 0$. On the other hand, the diagonal fields satisfy the following important relations

$$\widetilde{A}(\lambda)\left|0\right\rangle^{(\pm)} = \prod_{j=1}^{L}\left(\lambda+\frac{\eta}{2}+\eta S_j\right)\frac{y_1}{y_{L+1}}\left|0\right\rangle^{(\pm)}, \tag{22}$$

$$\widetilde{D}(\lambda)\left|0\right\rangle^{(\pm)} = \prod_{j=1}^{L}\left(\lambda+\frac{\eta}{2}-\eta S_j\right)\frac{y_{L+1}}{y_1}\frac{\det\left(M_1\right)}{\det\left(M_{L+1}\right)}\left|0\right\rangle^{(\pm)}. \tag{23}$$

From these properties and the help of Eq.(20) it is not difficult to see that the states $\left|0\right\rangle^{(\pm)}$ are themselves an eigenstate of $T_{1/2}^{(S_1,\dots,S_L)}(\lambda)$ with eigenvalue $\Lambda_{0\pm}^{(S_1,\dots,S_L)}(\lambda)$ given by

$$\Lambda_{0_\pm}^{(S_1,\dots,S_L)}(\lambda) = g_{1/2}^{(\pm)}\prod_{j=1}^{L}\left(\lambda+\frac{\eta}{2}+\eta S_j\right)+g_{1/2}^{(\mp)}\prod_{j=1}^{L}\left(\lambda+\frac{\eta}{2}-\eta S_j\right), \tag{24}$$

where the phase factors $g_{1/2}^{(\pm)}$ are the eigenvalues of the matrix $\mathcal{G}_{\mathcal{A}}$

$$g_{1/2}^{(\pm)} = \frac{(g_{11}+g_{22})\pm\sqrt{(g_{11}-g_{22})^2+4g_{12}g_{21}}}{2}. \tag{25}$$

This result strongly suggest us that the arbitrary eigenvectors $\left|\phi\right\rangle^{(\pm)}$ of $T_{1/2}^{(S_1,\dots,S_L)}(\lambda)$ can be put in the form

$$\left|\phi\right\rangle^{(\pm)} = \prod_{j=1}^{n_\pm}\widetilde{B}(\lambda_j^{(\pm)})\left|0\right\rangle^{(\pm)}. \tag{26}$$

The symmetry $\left[\check{R}(\lambda), M_j\otimes M_j\right]=0$ implies that the gauge transformed monodromy matrix $\widetilde{\mathcal{T}_{\mathcal{A}}}(\lambda)$ satisfies the Yang-Baxter algebra (1) with the same R-matrix as the original monodromy matrix $\mathcal{T}_{\mathcal{A}}(\lambda)$. As consequence of that, the matrix elements of $\widetilde{\mathcal{T}_{\mathcal{A}}}(\lambda)$ satisfies the same set of commutation rules of the periodic six-vertex model [1, 2] and from now on the main steps in the eigenvalue solution of $T_{1/2}^{(S_1,\dots,S_L)}(\lambda)$ become very similar to that of this well known vertex system. Considering that these details have appeared in many different places in the literature, see for instance ref. [1, 2], here we shall present only the final results. By imposing that $\left|\phi\right\rangle^{(\pm)}$ are eigenstates of $T_{1/2}^{(S_1,\dots,S_L)}(\lambda)$ and performing the convenient displacement $\lambda_i^{(\pm)}\rightarrow\lambda_i^{(\pm)}-\frac{\eta}{2}$

we find that the corresponding eigenvalues $\Lambda_{n_\pm}^{(S_1,\dots,S_L)}(\lambda)$ are given by the expression

$$\begin{aligned}\Lambda_{n_\pm}^{(S_1,\dots,S_L)}(\lambda) &= g_{1/2}^{(\pm)} \prod_{j=1}^{L} \left(\lambda + \frac{\eta}{2} + \eta S_j\right) \prod_{i=1}^{n_\pm} \frac{\lambda_i^{(\pm)} - \lambda + \frac{\eta}{2}}{\lambda_i^{(\pm)} - \lambda - \frac{\eta}{2}} \\ &\quad + g_{1/2}^{(\mp)} \prod_{j=1}^{L} \left(\lambda + \frac{\eta}{2} - \eta S_j\right) \prod_{i=1}^{n_\pm} \frac{\lambda - \lambda_i^{(\pm)} + \frac{3\eta}{2}}{\lambda - \lambda_i^{(\pm)} + \frac{\eta}{2}}, \end{aligned} \tag{27}$$

where the rapidities $\lambda_i^{(\pm)}$ satisfy the following system of transcendental equations

$$\prod_{j=1}^{L} \left(\frac{\lambda_i^{(\pm)} + \eta S_j}{\lambda_i^{(\pm)} - \eta S_j} \right) = \frac{g_{1/2}^{(\mp)}}{g_{1/2}^{(\pm)}} \prod_{\substack{l=1 \\ l \neq i}}^{n_\pm} \frac{\lambda_i^{(\pm)} - \lambda_l^{(\pm)} + \eta}{\lambda_i^{(\pm)} - \lambda_l^{(\pm)} - \eta}. \tag{28}$$

We would like to close this section with the following comments. First we emphasize that the integers $n_\pm$ play the role of particle numbers sectors satisfying the constraint $n_\pm \leq 2\sum_{i=1}^{L} S_i$. In appendix A, we illustrate this fact by presenting the details of a study of the completeness of the eigenspectrum (27,28) for $L = 2$. Our final observation concerns with physically interesting spin chains that commutes with the transfer matrix (10). Though the integrability does not depend on how we distribute the $\mathcal{L}$-operators $\mathcal{L}_{\mathcal{A}j}^{(\frac{1}{2},S_j)}(\lambda)$, the construction of local conserved charges commuting with $T_{1/2}^{(S_1,\dots,S_L)}(\lambda)$ does. One interesting case is when we have only one impurity $\mathcal{L}$-operator $\mathcal{L}_{\mathcal{A}L}^{(\frac{1}{2},S)}(\lambda)$ sitting at the end of the chain of $L-1$ spin-1/2 $\mathcal{L}_{\mathcal{A}j}^{(\frac{1}{2},\frac{1}{2})}(\lambda)$ Boltzmann weights. Because $\mathcal{L}_{\mathcal{A}j}^{(\frac{1}{2},\frac{1}{2})}(\lambda)$ is proportional to the exchange operator $P_{\mathcal{A}j}$ at the point $\lambda = 0$, we can produce local charges by expanding $\ln\left[T_{1/2}^{(\frac{1}{2},\dots,\frac{1}{2},S)}(\lambda)\right]$ around this point. The second term in this expansion is the associated Hamiltonian and its general expression reads

$$\mathcal{H} = \frac{1}{\eta}\sum_{i=1}^{L-2} \left(\mathcal{L}_{i,i+1}^{(\frac{1}{2},\frac{1}{2})}(0)\right)^{-1} + \left(\mathcal{L}_{L-1,L}^{(\frac{1}{2},S)}(0)\right)^{-1} + \frac{1}{\eta}\left(\mathcal{L}_{L-1,L}^{(\frac{1}{2},S)}(0)\right)^{-1} \mathcal{G}_{L-1}^{-1} \left(\mathcal{L}_{L-1,1}^{(\frac{1}{2},\frac{1}{2})}(0)\right)^{-1} \mathcal{G}_{L-1} \mathcal{L}_{L-1,L}^{(\frac{1}{2},S)}(0), \tag{29}$$

where we have implicitly assumed that the boundary matrix $\mathcal{G}_{\mathcal{A}}$ is non singular.

By substituting the expression for the weights $\mathcal{L}_{\mathcal{A}j}^{(\frac{1}{2},S_j)}(\lambda)$ in the above equation and after

several manipulations we find that $\mathcal{H}$ can be written as

$$\mathcal{H} = \frac{2}{\eta}\sum_{i=1}^{L-2}\sum_{\alpha=1}^{3}\sigma_i^\alpha\sigma_{i+1}^\alpha + \frac{1}{2\eta}\left(L-1+\frac{1}{(S+\frac{1}{2})^2}\right) + \frac{2}{\eta}\left(\frac{1}{S+\frac{1}{2}}\right)^2 \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{3}\sigma_{L-1}^\alpha\left\{S_L^\alpha, S_L^\beta\right\}\sigma_{L+1}^\beta$$

$$\frac{2}{\eta}\left(\frac{1}{S+\frac{1}{2}}\right)^2\sum_{\alpha=1}^{3}\left[\sigma_{L-1}^\alpha S_L^\alpha + S_L^\alpha\sigma_{L+1}^\alpha + 2\sigma_{L-1}^\alpha(S_L^\alpha)^2\sigma_{L+1}^\alpha + \left(\frac{1}{4}-S(S+1)\right)\sigma_{L-1}^\alpha\sigma_{L+1}^\alpha\right] \quad (30)$$

where the index $\alpha = 1, 2, 3$ means the angular momenta components x, y, z, respectively. In the expression (30) is also assumed the following toroidal boundary condition between the sites 1 and $L+1$ [11]

$$\begin{pmatrix} \sigma_{L+1}^+ \\ \sigma_{L+1}^- \\ \sigma_{L+1}^z \end{pmatrix} = \frac{1}{g_{11}g_{22}-g_{12}g_{21}}\begin{pmatrix} g_{11}^2 & -g_{21}^2 & -g_{11}g_{21} \\ -g_{12}^2 & g_{22}^2 & g_{12}g_{22} \\ -2g_{11}g_{12} & 2g_{21}g_{22} & g_{11}g_{22}+g_{12}g_{21} \end{pmatrix}\begin{pmatrix} \sigma_1^+ \\ \sigma_1^- \\ \sigma_1^z \end{pmatrix}. \quad (31)$$

We see that the boundary matrix $\mathcal{G}_\mathcal{A}$ permits a more general type of interactions between the bulk spin-1/2 and the spin-S impurity. The model (30,31) turns out to be an interesting extension of the impurity system originally proposed in ref.[12]. Finally, the eigenvalues $E^{(\pm)} = \left[\frac{\mathrm{d}}{\mathrm{d}\,\lambda}\ln\left(\Lambda_{n_\pm}^{(\frac{1}{2},\dots,\frac{1}{2},S)}(\lambda)\right)\right]|_{\lambda=0}$ of this Hamiltonian are given by

$$E^{(\pm)} = \frac{1}{\eta}\left[L-1+\frac{1}{S+\frac{1}{2}}+\eta^2\sum_{i=1}^{n_\pm}\frac{1}{[\lambda_i^{(\pm)}]^2-\frac{\eta^2}{4}}\right], \quad (32)$$

where $\lambda_i^{(\pm)}$ satisfy the same Bethe ansatz equation (28).

2.2 The non-compact $SU(1,1)$ realization

Another possible realization of the Yang-Baxter algebra (1) for the $SU(2)$ R-matrix is in terms of the $SU(1,1)$ Lie algebra. One motivation to study integrable models whose quantum space are $SU(1,1)$ invariant comes from the physics of atoms interacting with electromagnetic fields. In fact, different realizations of this symmetry in terms of bosonic operators lead us to a number of solvable atom-fields models [15, 16] that are interesting generalizations of the famous Jaynes and Cummings paradigm [17]. In this context, the study of these systems with

general boundary representations appears to be important, since recently it has been argued [18] that suitable combinations between boundary matrices $\mathcal{G}_{\mathcal{A}}$ and $\mathcal{L}$-operators can generated the so-called counter-rotating terms which are relevant in the case of high intensity fields.

The $\mathcal{L}$-operator realization in terms of the $SU(1,1)$ generators K_j^z and $K_j^{\pm}$ is given by [19],

$$\mathcal{L}_{\mathcal{A}j}^{(\frac{1}{2},k_j)}(\lambda) = \begin{pmatrix} [\lambda + c] + \eta K_j^z & -\eta K_j^- \\ \eta K_j^+ & [\lambda + c] - \eta K_j^z \end{pmatrix}, \tag{33}$$

where k_j is the Bargmann index [20] which characterize the unitary representations of the $SU(1,1)$ algebra and c is any complex constant. The commuting transfer matrix of the corresponding mixed vertex model is therefore,

$$T_{1/2}^{(k_1,\dots,k_L)}(\lambda) = \mathrm{Tr}_{\mathcal{A}}\left[\mathcal{G}_{\mathcal{A}}\mathcal{L}_{\mathcal{A}L}^{(\frac{1}{2},k_L)} \dots \mathcal{L}_{\mathcal{A}1}^{(\frac{1}{2},k_1)}\right]. \tag{34}$$

The spectrum of this operator will depend much on the kind of representation we choose for the quantum space. One relevant representation for our purposes is the positive discrete series $D^+(k_j)$ where k_j assume integer or half-integer values. More specifically, let the ket $|k_j, n_j\rangle$, with $n_j = 0, 1, 2, \dots$ be the basis of $D^+(k_j)$, then the action of the generators is given by

$$K_j^z\,|k_j, n_j\rangle = (n_j + k_j)\,|k_j, n_j\rangle\,, \tag{35a}$$

$$K_j^+\,|k_j, n_j\rangle = \sqrt{(n_j+1)(n_j+2k_j)}\,|k_j, n_j+1\rangle\,, \tag{35b}$$

$$K_j^-\,|k_j, n_j\rangle = \sqrt{n_j(n_j+2k_j-1)}\,|k_j, n_j-1\rangle\,. \tag{35c}$$

As before, we should now look for a local reference state $|0\rangle_j^{(k_j)} \in SU(1,1)$ such that the action of the transformed $M_{j+1}^{-1}\mathcal{L}_{\mathcal{A}j}^{(\frac{1}{2},k_j)}(\lambda)M_j$ on it gives as a result an up triangular matrix. This condition implies the following annihilation property

$$\left\{[\lambda + c]\,(p_{j+1} - p_j) - \eta K_j^z(p_{j+1} + p_j) + \eta K_j^+ p_{j+1}p_j + \eta K_j^-\right\}|0\rangle_j^{(k_j)} = 0. \tag{36}$$

In order to solve Eq.(36) we shall repeat the procedure of previous section, i.e. we expand the pseudovacuum $|0\rangle_j^{(k_j)} = \sum_{n_j=0}^{\infty} \bar{C}(k_j, n_j)\,|k_j, n_j\rangle$ in the $SU(1,1)$ basis and use the properties (35a-35c) as well as that p_j is the same for all j to generate relations for the coefficients

$\bar{C}(k_j, n_j)$. These relations are disentangled recursively and we find that the appropriate state $|0\rangle_j^{(k_j)}$ is

$$|0\rangle_j^{(k_j)} = \sum_{n_j=0}^{\infty} p_j^{n_j} \sqrt{\frac{(2k_j + n_j - 1)!}{(2k_j - 1)! n_j!}} \bar{C}(k_j, 0) \, |k_j, n_j\rangle \,, \tag{37}$$

and that the action of the $\mathcal{L}$-operator on it is given by

$$\widetilde{\mathcal{L}}_{\mathcal{A}j}^{(\frac{1}{2},k_j)}(\lambda) \, |0\rangle_j^{(k_j)} = \begin{pmatrix} (\lambda + c - \eta k_j) \, \frac{y_j}{y_{j+1}} \, |0\rangle_j^{(k_j)} & \# \\ 0 & (\lambda + c + \eta k_j) \, \frac{y_{j+1}}{y_j} \frac{\det(M_j)}{\det(M_{j+1})} \, |0\rangle_j^{(k_j)} \end{pmatrix}. \tag{38}$$

Before proceeding it is important to remark that the norm of $|0\rangle_j^{(k_j)}$ is $|\bar{C}(k_j, 0)|^2 (1 - p_j^2)^{-2k_j}$ which means that this state can be normalized except when $p_j = \pm 1$. This provides an extra requirement to the entries of the boundary matrix due to the constraint (20). Apart from this fact the next steps to determine the eigenvalues and eigenfunctions of $T_{1/2}^{(k_1,\dots,k_L)}(\lambda)$ by the quantum inverse scattering method are fairly parallel to those already described in section 2.1. Omitting these details the final result for the eigenvalues is

$$\begin{aligned} \Lambda_{n_\pm}^{(k_1,\dots,k_L)} \quad &= \quad g_{1/2}^{(\pm)} \prod_{j=1}^{L} (\lambda + c - \eta k_j) \prod_{i=1}^{n_\pm} \frac{\lambda_i^{(\pm)} - \lambda + \eta - c}{\lambda_i^{(\pm)} - \lambda - c} \\ &\quad + g_{1/2}^{(\mp)} \prod_{j=1}^{L} (\lambda + c + \eta k_j) \prod_{i=1}^{n_\pm} \frac{\lambda - \lambda_i^{(\pm)} + \eta + c}{\lambda - \lambda_i^{(\pm)} + c}, \end{aligned} \tag{39}$$

provided that the rapidities satisfy the Bethe ansatz equations

$$\prod_{j=1}^{L} \left(\frac{\lambda_i^{(\pm)} - \eta k_j}{\lambda_i^{(\pm)} + \eta k_j} \right) = \frac{g_{1/2}^{(\mp)}}{g_{1/2}^{(\pm)}} \prod_{\substack{l=1 \\ l \neq i}}^{n_\pm} \frac{\lambda_i^{(\pm)} - \lambda_l^{(\pm)} + \eta}{\lambda_i^{(\pm)} - \lambda_l^{(\pm)} - \eta}, \tag{40}$$

where we have performed the convenient shift $\lambda_i^{(\pm)} \to \lambda_i^{(\pm)} - c$.

We would like to finished this section with the following comment. Having at hand two different families of $\mathcal{L}$-operators $\mathcal{L}_{\mathcal{A}j}^{(k)}(\lambda) \in \prod_{j=1}^{L_k} \otimes V_j^{(k)}$ $k = 1, 2$ which satisfy the Yang-Baxter algebra with the same R-matrix, the co-multiplication structure of this algebra allows us to construct even more generalized mixed vertex models by combining the tensor product of these two possible realizations. The corresponding monodromy matrix can be written as

$$\mathcal{T}_{\mathcal{A}}(\lambda) = \mathcal{G}_{\mathcal{A}} \overline{\mathcal{L}}_{\mathcal{A}L_1+L_2}(\lambda) \overline{\mathcal{L}}_{\mathcal{A}L_1+L_2-1}(\lambda) \dots \overline{\mathcal{L}}_{\mathcal{A}2}(\lambda) \overline{\mathcal{L}}_{\mathcal{A}1}(\lambda), \tag{41}$$

where $\overline{\mathcal{L}}_{\mathcal{A}j}(\lambda)$ is defined by

$$\overline{\mathcal{L}}_{\mathcal{A}j}(\lambda) = \begin{cases} \mathcal{L}^{(1)}_{\mathcal{A}j}(\lambda), & \text{if } j \in \{\gamma_1, \dots, \gamma_{L_1}\} \\ \mathcal{L}^{(2)}_{\mathcal{A}j}(\lambda), & \text{otherwise,} \end{cases} \tag{42}$$

and the partition $\{\gamma_1, \dots, \gamma_L\}$ denotes a set of integer indices assuming values in the interval $1 \leq \gamma_i \leq L_1 + L_2$.

For example, one choose as the first family of $\mathcal{L}$-operators the higher spin operators (9) and as the second one the $SU(1,1)$ realization (33). Clearly, the eigenvalues problem associated with the general monodromy matrix (41) is solvable by trivial combination of the results of section 2. The global reference state $|0\rangle^{(\pm)}$ turns out to be[1]

$$|0\rangle^{(\pm)} = \prod_{j \in \{\beta_1, \dots, \beta_L\}} \otimes |0\rangle_j^{(S_j)} \prod_{\substack{j \\ \text{otherwise}}} \otimes |0\rangle_j^{(k_j)}, \tag{43}$$

while the eigenvalues are

$$\begin{aligned} \Lambda_{n_\pm}^{(\{S_j\},\{k_j\})}(\lambda) = {} & g_{1/2}^{(\pm)} \prod_{j=1}^{L_1} \left(\lambda + \frac{\eta}{2} + \eta S_j\right) \prod_{j=1}^{L_2} (\lambda + c - \eta k_j) \prod_{i=1}^{n_\pm} \frac{\lambda_i^{(\pm)} - \lambda + \eta}{\lambda_i^{(\pm)} - \lambda} \\ & + g_{1/2}^{(\mp)} \prod_{j=1}^{L_1} \left(\lambda + \frac{\eta}{2} - \eta S_j\right) \prod_{j=1}^{L_2} (\lambda + c + \eta k_j) \prod_{i=1}^{n_\pm} \frac{\lambda - \lambda_i^{(\pm)} + \eta}{\lambda - \lambda_i^{(\pm)}}, \end{aligned} \tag{44}$$

and $\lambda_i^{\pm}$ satisfies the Bethe ansatz equation

$$\prod_{j=1}^{L_1} \left(\frac{\lambda_i^{(\pm)} + \frac{\eta}{2} + \eta S_j}{\lambda_i^{(\pm)} + \frac{\eta}{2} - \eta S_j} \right) \prod_{j=1}^{L_2} \left(\frac{\lambda_i^{(\pm)} + c - \eta k_j}{\lambda_i^{(\pm)} + c + \eta k_j} \right) = \frac{g_{1/2}^{(\mp)}}{g_{1/2}^{(\pm)}} \prod_{\substack{l=1 \\ l \neq i}}^{n_\pm} \frac{\lambda_i^{(\pm)} - \lambda_l^{(\pm)} + \eta}{\lambda_i^{(\pm)} - \lambda_l^{(\pm)} - \eta}. \tag{45}$$

3 $SU(N)$ mixed vertex model

A natural extension of previous section would be to consider mixed vertex models whose Boltzmann weights that combines isomorphic and non-isomorphic $\mathcal{L}$-operators based on the

[1] We recall that here we are supposing that the boundary elements respect the condition $p_j \neq \pm 1$ to make the state $|0\rangle_j^{(k_j)}$ normalizable.

$SU(N)$ symmetry at different orders of representations. Previous results for this mixed model [21], however, suggest us that the details entering the solution of such systems with general twists will follow closely that already described in section 2.1 and in our previous work [11].

However, because $SU(N)$ is not self-conjugate for $N \geq 3$ one expects that the Yang-Baxter algebra (1) admits further classes of realization on this group other than the higher spin representations. The purpose of this section is to explore such possibility by considering $\mathcal{L}$-operators whose quantum space is invariant by the conjugate representation of $SU(N)$. Originally this $\mathcal{L}$-operator was discovered in the context of factorizable theories, representing the scattering matrix between particles and anti-particles [22]. Recently [23], this amplitude has been discussed more generally in terms of the braid-monoid algebra. We also recall that mixed vertex model combining the fundamental and conjugate representations are of direct interest in statistical mechanics since it appears to be related with the combinatorial problem of coloring the edges of the square lattice and fully packed loop models [24]. The commuting transfer matrix $T^{(L_1,L_2)}(\lambda)$ of such mixed vertex model with a boundary $\mathcal{G}_{\mathcal{A}}$ is defined by Eqs.(3,41). The first $\mathcal{L}^{(1)}_{\mathcal{A}j}(\lambda)$ operator is the fundamental $SU(N)$ realization obtained from Eqs.(5,7), namely

$$\mathcal{L}^{(1)}_{\mathcal{A}j}(\lambda) = \lambda \sum_{\alpha=1}^{N} \hat{e}^{(a)}_{\alpha\alpha} \otimes \hat{e}^{(b)}_{\alpha\alpha} + \eta \sum_{\alpha,\beta=1}^{N} \hat{e}^{(a)}_{\alpha\beta} \otimes \hat{e}^{(b)}_{\beta\alpha}, \tag{46}$$

while the second one $\mathcal{L}^{(2)}_{\mathcal{A}j}(\lambda)$ intertwines between the fundamental and conjugate representation of $SU(N)$ and its expression is [23]

$$\mathcal{L}^{(2)}_{\mathcal{A}j}(\lambda) = \left(\frac{\lambda}{\eta} - \rho\right) \sum_{\alpha=1}^{N} \hat{e}^{(a)}_{\alpha,\alpha} \otimes \hat{e}^{(b)}_{\alpha,\alpha} - \sum_{\alpha,\beta=1}^{N} \hat{e}^{(a)}_{\alpha,\beta} \otimes \hat{e}^{(b)}_{N+1-\alpha,N+1-\beta}. \tag{47}$$

where ρ is an extra free parameter.

Our first task is to find the reference states for each transformed operator $\widetilde{\mathcal{L}}^{(i)}_{\mathcal{A}j}(\lambda) = M^{-1}_{j+1}\mathcal{L}^{(i)}_{\mathcal{A}j}(\lambda)M_j$ that annihilates all the $\frac{N(N-1)}{2}$ lower left elements for arbitrary λ. The form of such pseudovacuums has a direct dependence on the elements of the gauge matrix which here will be denoted by $M_j = \sum_{\alpha,\beta=1}^{N} m_j(\alpha,\beta)\hat{e}^{(\mathcal{A})}_{\alpha\beta}$. The first local reference state $|0\rangle^{(1)}_j$ has been

already determined in ref.[11] and it is given by the first column of the gauge matrix M_j, namely

$$|0\rangle_j^{(1)} = \begin{pmatrix} m_j(1,1) \\ m_j(2,1) \\ \vdots \\ m_j(N-1,1) \\ m_j(N,1) \end{pmatrix}_j , \tag{48}$$

provided that the following ratios are satisfied

$$p_{\alpha,\beta} = \frac{m_j(\alpha,\beta)}{m_j(N,\beta)} = \frac{m_{j+1}(\alpha,\beta)}{m_{j+1}(N,\beta)}, \quad \text{for} \quad \alpha,\beta = 1,\dots,N-1. \tag{49}$$

The second reference state $|0\rangle_j^{(2)}$ has a more involved representation in terms of the elements of M_j. It turns out that this reference state can be put in the following form

$$|0\rangle_j^{(2)} = \begin{pmatrix} c_{N,N}^{(j)} \\ c_{N-1,N}^{(j)} \\ \vdots \\ c_{2,N}^{(j)} \\ c_{1,N}^{(j)} \end{pmatrix}_j , \tag{50}$$

where $c_{\alpha,\beta}^{(j)}$ are the cofactors of the gauge matrix M_j. These are simply obtained from the (α,β) minors of M_j with an appropriate sign,

$$c_{\alpha,\beta}^{(j)} = (-1)^{\alpha+\beta} \begin{vmatrix} m_j(1,1) & \dots & m_j(1,\beta-1) & m_j(1,\beta+1) & \dots & m_j(1,N) \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ m_j(\alpha-1,1) & \dots & m_j(\alpha-1,\beta-1) & m_j(\alpha-1,\beta+1) & \dots & m_j(\alpha-1,N) \\ m_j(\alpha+1,1) & \dots & m_j(\alpha+1,\beta-1) & m_j(\alpha+1,\beta+1) & \dots & m_j(\alpha+1,N) \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ m_j(N,1) & \dots & m_j(N,\beta-1) & m_j(N,\beta+1) & \dots & m_j(N,N) \end{vmatrix} . \tag{51}$$

It follows that the action of the transformed $\mathcal{L}$-operator for fundamental and conjugate representation on their respective reference state is given in terms of an up triangular matrix,

$$\widetilde{\mathcal{L}}^{(i)}_{\mathcal{A}j}(\lambda)\,|0\rangle^{(i)}_j = \begin{pmatrix} a^{(i)}_1(\lambda)\frac{f^j_1}{f^{j+1}_1} & \# & \cdots & \# & \# \\ 0 & a^{(i)}_2(\lambda)\frac{f^j_2}{f^{j+1}_2} & \cdots & \# & \# \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a^{(i)}_{N-1}(\lambda)\frac{f^j_{N-1}}{f^{j+1}_{N-1}} & \# \\ 0 & 0 & \cdots & 0 & a^{(i)}_N(\lambda)\frac{f^j_N}{f^{j+1}_N} \end{pmatrix}_{N\times N} |0\rangle^{(i)}_j, \quad (52)$$

where f^j_α are given by

$$f^j_\alpha = \begin{cases} m_j(N,\alpha), \quad \alpha = 1,\dots,N-1 \\ \left(\prod_{i=1}^{N-1}\frac{1}{m_j(N,i)}\right)\det(M_j), \quad \alpha = N, \end{cases} \quad (53)$$

while the rational functions $a^{(i)}_k(\lambda)$ are

$$a^{(1)}_k(\lambda) = \begin{cases} \lambda+\eta \;\text{ for }\; k=1 \\ \lambda \;\text{ for }\; k=2,\dots,N \end{cases} \qquad a^{(2)}_k(\lambda) = \begin{cases} \frac{\lambda}{\eta}-\rho \;\text{ for }\; k=1,\dots,N-1 \\ \frac{\lambda}{\eta}-\rho-1 \;\text{ for }\; k=N. \end{cases} \quad (54)$$

The next step is to use the the remaining freedom of the elements of the gauge matrix to transform $M_1^{-1}\mathcal{G}_{\mathcal{A}}M_{L+1}$ into a diagonal matrix. This condition together with Eq.(49) impose severe restrictions on the possible values for the ratios $p_{\alpha,\beta}$ which turns out to be the same satisfied by the ratio of the components of the eigenvectors of the boundary matrix $\mathcal{G}_{\mathcal{A}}$. This results in N possible choices for $p^{(l)}_{\alpha,1}$, $l=1,\dots,N$ and therefore from Eqs.(48,50) there exists N type of appropriate local references states that will be denoted by $|0\rangle^{(i,l)}_j$. As a consequence of that we have N possible choices for the global pseudovacuum which can be written as

$$|0\rangle^{(l)} = \prod_{j\in\{\beta_1,\dots,\beta_L\}} \otimes\, |0\rangle^{(1,l)}_j \prod_{\substack{j \\ \text{otherwise}}} \otimes\, |0\rangle^{(2,l)}_j \quad \text{for }\; l=1,\dots,N. \quad (55)$$

Further progresses are made by choosing a suitable representation for the gauge transformed monodromy matrix that are able to distinguish the creation and annihilation fields over the

reference state (55). Previous experience with vertex models having the triangular property (52) suggest us to take the structure used in the nested Bethe ansatz formulation of periodic $SU(N)$ models [9, 10]

$$\widetilde{\mathcal{T}_{\mathcal{A}}}(\lambda) = \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}_1(\lambda) & \cdots & \widetilde{B}_{N-1}(\lambda) \\ \widetilde{C}_1(\lambda) & \widetilde{D}_{11}(\lambda) & \cdots & \widetilde{D}_{1N-1}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{C}_{N-1}(\lambda) & \widetilde{D}_{N-11}(\lambda) & \cdots & \widetilde{D}_{N-1N-1}(\lambda) \end{pmatrix}_{N\times N} , \tag{56}$$

By comparing Eq.(52) with Eq.(56) we observe that the operators $\widetilde{B}_k(\lambda)$ produce new states when applied to the pseudovacuum state $|0\rangle^{(l)}$ while $\widetilde{C}_k(\lambda)$ are clearly annihilation fields. Furthermore, we can also read the action of the diagonal fields,

$$\widetilde{A}(\lambda)\,|0\rangle^{(l)} = [a_1^{(1)}(\lambda)]^{L_1}[a_1^{(2)}(\lambda)]^{L_2}\frac{f_1^1}{f_1^{L+1}}\,|0\rangle^{(l)}\,, \tag{57}$$

$$\widetilde{D}_{kk}(\lambda)\,|0\rangle^{(l)} = [a_{k+1}^{(1)}(\lambda)]^{L_1}[a_{k+1}^{(2)}(\lambda)]^{L_2}\frac{f_{k+1}^1}{f_{k+1}^{L+1}}\,|0\rangle^{(l)}\,, \quad k = 1,\dots,N-1. \tag{58}$$

We now seek for further eigenstates $|\phi\rangle^{(l)}$ of $T^{(L_1,L_2)}(\lambda)$ with the following structure

$$|\phi\rangle^{(l)} = \widetilde{B}_{a_1}(\lambda_1^{(1,l)})\dots\widetilde{B}_{a_{m_1^{(l)}}}(\lambda_{m_1^{(l)}}^{(1,l)})\mathcal{F}^{a_{m_1^{(l)}}\dots a_1}\,|0\rangle^{(l)}\,, \tag{59}$$

where the indices a_j of the coefficients $\mathcal{F}^{a_{m_1^{(l)}}\dots a_1}$ run over $N-1$ possible values. The rapidities $\lambda_j^{(1,l)}$ will be determined by solving the eigenvalue equation

$$T^{(L_1,L_2)}(\lambda)\,|\phi\rangle^{(l)} = \Lambda^{(L_1,L_2)}(\lambda)\,|\phi\rangle^{(l)}\,. \tag{60}$$

At this point we have the basic ingredients to follow the general strategy of the nested Bethe ansatz method. We apply either $\widetilde{A}(\lambda)$ or $\widetilde{D}_{kk}(\lambda)$ on $|\phi\rangle^{(l)}$ with the help of commutation rules that are the same known for the periodic case as explained in our previous work [11]. From now on the computations are standard and since the few adaptations needed have already been described in ref.[11] we shall present here only the final results. We find that the eigenvalues

$\Lambda^{(L_1,L_2)}_{m_1^{(l)}\ \dots\ m_{N-1}^{(l)}}(\lambda)$ are given by

$$
\begin{aligned}
\Lambda^{(L_1,L_2)}_{m_1^{(l)}\ \dots\ m_{N-1}^{(l)}}(\lambda;\{\lambda_i^{(1,l)}\},\dots,\{\lambda_i^{(N-1,l)}\}) &= g^{(l)}\,(\lambda+\eta)^{L_1}\left(\frac{\lambda}{\eta}-\rho\right)^{L_2}\prod_{j=1}^{m_1^{(l)}}\frac{\lambda_j^{(1,l)}-\lambda+\eta}{\lambda_j^{(1,l)}-\lambda} \\
+(\lambda)^{L_1}\left(\frac{\lambda}{\eta}-\rho\right)^{L_2}\sum_{k=1}^{N-2}g^{(l+k)}\prod_{j=1}^{m_k^{(l)}}\frac{\lambda-\lambda_j^{(k,l)}+\eta}{\lambda-\lambda_j^{(k,l)}}\prod_{j=1}^{m_{k+1}^{(l)}}\frac{\lambda_j^{(k+1,l)}-\lambda+\eta}{\lambda_j^{(k+1,l)}-\lambda} \\
+(\lambda)^{L_1}\left(\frac{\lambda}{\eta}-\rho-1\right)^{L_2}g^{(l+N-1)}\prod_{j=1}^{m_{N-1}^{(l)}}\frac{\lambda-\lambda_j^{(N-1,l)}+\eta}{\lambda-\lambda_j^{(N-1,l)}}, \qquad (61)
\end{aligned}
$$

where the phase factors $g^{(l)}$ are the eigenvalues of the boundary gauge matrix $\mathcal{G}_{\mathcal{A}}$ ordered in such way that they satisfy the relation $g^{(l)}=g^{(l+N)}$ for $l=1,\dots,N$. The corresponding nested Bethe ansatz equations can be written as

$$
\frac{g^{(l)}}{g^{(l+1)}}\left[\frac{\lambda_i^{(1,l)}+\eta}{\lambda_i^{(1,l)}}\right]^{L_1}=\prod_{\substack{j=1\\ j\neq i}}^{m_1^{(l)}}-\frac{\lambda_i^{(1,l)}-\lambda_j^{(1,l)}+\eta}{\lambda_j^{(1,l)}-\lambda_i^{(1,l)}+\eta}\prod_{j=1}^{m_2^{(l)}}\frac{\lambda_j^{(2,l)}-\lambda_i^{(1,l)}+\eta}{\lambda_j^{(2,l)}-\lambda_i^{(1,l)}}, \qquad (62)
$$

$$
\frac{g^{(l+k-1)}}{g^{(l+k)}}\prod_{j=1}^{m_{k-1}^{(l)}}\frac{\lambda_i^{(k,l)}-\lambda_j^{(k-1,l)}+\eta}{\lambda_i^{(k,l)}-\lambda_j^{(k-1,l)}}=\prod_{\substack{j=1\\ j\neq i}}^{m_k^{(l)}}-\frac{\lambda_i^{(k,l)}-\lambda_j^{(k,l)}+\eta}{\lambda_j^{(k,l)}-\lambda_i^{(k,l)}+\eta}\prod_{j=1}^{m_{k+1}^{(l)}}\frac{\lambda_j^{(k+1,l)}-\lambda_i^{(k,l)}+\eta}{\lambda_j^{(k+1,l)}-\lambda_i^{(k,l)}}, \qquad (63)
$$

$$
k=2,\dots,N-2
$$

$$
\frac{g^{(l+N-2)}}{g^{(l+N-1)}}\left[\frac{\lambda_i^{(N-1,l)}-\eta\rho}{\lambda_i^{(N-1,l)}-\eta(\rho-1)}\right]^{L_2}\prod_{j=1}^{m_{N-2}^{(l)}}\frac{\lambda_i^{(N-1,l)}-\lambda_j^{(N-2,l)}+\eta}{\lambda_i^{(N-1,l)}-\lambda_j^{(N-2,l)}}=\prod_{\substack{j=1\\ j\neq i}}^{m_{N-1}^{(l)}}-\frac{\lambda_i^{(N-1,l)}-\lambda_j^{(N-1,l)}+\eta}{\lambda_j^{(N-1,l)}-\lambda_i^{(N-1,l)}+\eta}. \qquad (64)
$$

4 Quantum space transformations

One striking result of previous sections is that the final forms of the transfer matrix eigenvalues and the Bethe ansatz equations are similar to that expected for vertex models with diagonal

boundaries. The corresponding diagonal twists are just the eigenvalues of the non-diagonal boundary matrix $\mathcal{G}_{\mathcal{A}}$. This result motive us to look for possible correspondences between the eigenvectors of such vertex models with non-diagonal and diagonal boundaries as well.

In order to establish the above mentioned relationship one has to explore the possibility of making quantum space transformations on the gauge transformed $\widetilde{\mathcal{L}}$-operators. Rather remarkably, for all the vertex models discussed in this paper it is always possible to choose a invertible transformations U_j on the space V_j such that

$$U_j^{-1}\widetilde{\mathcal{L}}_{\mathcal{A}j}(\lambda)U_j = \mathcal{L}_{\mathcal{A}j}(\lambda). \tag{65}$$

This means that quantum space transformations can be used to undo the modifications carried out on the $\mathcal{L}$-operators by the gauge transformations M_j. Let us now exemplify the importance of this property on the diagonalization of the standard transfer matrix defined by Eqs.(3,6). Denoting by $M_{\mathcal{A}}$ the matrix that diagonalize the boundary matrix $\mathcal{G}_{\mathcal{A}}$ we can write $T(\lambda)$ as

$$\begin{aligned} T(\lambda) &= \mathrm{Tr}_{\mathcal{A}}\left[M_{\mathcal{A}}D_{\mathcal{A}}M_{\mathcal{A}}^{-1}M_{\mathcal{A}}\left(M_{\mathcal{A}}^{-1}\mathcal{L}_{\mathcal{A}L}(\lambda)M_{\mathcal{A}}\right)\dots\left(M_{\mathcal{A}}^{-1}\mathcal{L}_{\mathcal{A}1}(\lambda)M_{\mathcal{A}}\right)M_{\mathcal{A}}^{-1}\right], && (66)\\ &= \mathrm{Tr}_{\mathcal{A}}\left[D_{\mathcal{A}}\widetilde{\mathcal{L}}_{\mathcal{A}L}(\lambda)\widetilde{\mathcal{L}}_{\mathcal{A}L-1}(\lambda)\dots\widetilde{\mathcal{L}}_{\mathcal{A}1}(\lambda)\right], && (67)\end{aligned}$$

where $D_{\mathcal{A}}$ is diagonal matrix whose entries are the eigenvalues of $\mathcal{G}_{\mathcal{A}}$ and the $\widetilde{\mathcal{L}}$-operators are given by

$$\widetilde{\mathcal{L}}_{\mathcal{A}j}(\lambda) = M_{\mathcal{A}}^{-1}\mathcal{L}_{\mathcal{A}j}(\lambda)M_{\mathcal{A}}. \tag{68}$$

Now motivated by property (65) we can define a new operator $T'(\lambda)$

$$T'(\lambda) = \prod_{j=1}^{L}\otimes U_j^{-1}T(\lambda)\prod_{j=1}^{L}\otimes U_j = \mathrm{Tr}_{\mathcal{A}}\left[D_{\mathcal{A}}\mathcal{L}_{\mathcal{A}L}(\lambda)\dots\mathcal{L}_{\mathcal{A}1}(\lambda)\right], \tag{69}$$

which is precisely the transfer matrix of the vertex model we have started with with diagonal twist $D_{\mathcal{A}}$.

At this point is important to recall that the transfer matrix $T'(\lambda)$ can be diagonalized with very little difference from the periodic case because the diagonal boundary $D_{\mathcal{A}}$ does not change

drastically the properties of the monodromy matrix elements. This not only explain the reason why the Bethe ansatz equations and eigenvalues have the same shape for both diagonal and non-diagonal boundaries but also makes it possible to substantiate a clear relation between their eigenvectors. In fact, if we denote by $|\psi'\rangle$ one possible eigenstate of $T'(\lambda)$ then it follows directly from (69) that the corresponding eigenkets $|\psi\rangle$ of $T(\lambda)$ will be given by

$$|\psi\rangle = \prod_{j=1}^{L} \otimes U_j \, |\psi\rangle \, , \tag{70}$$

and that the eigenvalues of $T'(\lambda)$ and $T(\lambda)$ are of course identical.

Explicit expressions for the quantum matrices U_j are summarized in appendix B for all the vertex models discussed so far. It turns out that the above observations can be used to diagonalize the transfer matrix of even more complicated mixed vertex models. For instance, let us consider the following operator

$$T_g(\lambda) = \mathrm{Tr}\left[{\mathcal{G}_{\mathcal{A}}}^{(L)} \mathcal{L}_{\mathcal{A}L}(\lambda) {\mathcal{G}_{\mathcal{A}}}^{(L-1)} \mathcal{L}_{\mathcal{A}L-1}(\lambda) \dots {\mathcal{G}_{\mathcal{A}}}^{(1)} \mathcal{L}_{\mathcal{A}1}(\lambda) \right], \tag{71}$$

that combines boundary matrices and $\mathcal{L}$-operators at any site of the lattice.

Assuming that ${\mathcal{G}_{\mathcal{A}}}^{(j)}$ are non-singular matrices one can insert ${\mathcal{G}_{\mathcal{A}}}^{(j)} \left[{\mathcal{G}_{\mathcal{A}}}^{(j)} \right]^{-1}$ terms all over the trace (71), permitting us to rewrite the transfer matrix $T_g(\lambda)$ as

$$T_g(\lambda) = \mathrm{Tr}\left[{\mathcal{G}_{\mathcal{A}}}^{(ef)} \widetilde{\mathcal{L}}_{\mathcal{A}L}(\lambda) \widetilde{\mathcal{L}}_{\mathcal{A}L-1}(\lambda) \dots \widetilde{\mathcal{L}}_{\mathcal{A}1}(\lambda) \right], \tag{72}$$

where the new boundary matrix ${\mathcal{G}_{\mathcal{A}}}^{(ef)}$ is given in terms of the following ordered product

$${\mathcal{G}_{\mathcal{A}}}^{(ef)} = {\mathcal{G}_{\mathcal{A}}}^{(L)} {\mathcal{G}_{\mathcal{A}}}^{(L-1)} \dots {\mathcal{G}_{\mathcal{A}}}^{(1)}. \tag{73}$$

The transformed $\widetilde{\mathcal{L}}$-operators in Eq.(72) are given by an extension of formula (68) that accommodates site dependent transformations, namely

$$\widetilde{\mathcal{L}}_{\mathcal{A}j}(\lambda) = \left[M_{\mathcal{A}}^{(j)} \right]^{-1} \mathcal{L}_{\mathcal{A}j}(\lambda) M_{\mathcal{A}}^{(j)}, \tag{74}$$

in such way that for each site j the matrices $M_{\mathcal{A}}^{(j)}$ are

$$M_{\mathcal{A}}^{(j)} = {\mathcal{G}_{\mathcal{A}}}^{(j-1)} {\mathcal{G}_{\mathcal{A}}}^{(j-2)} \dots {\mathcal{G}_{\mathcal{A}}}^{(1)}. \tag{75}$$

As before, the next step is to find the appropriate quantum transformations U_j in order to reverse the gauge transformations $M_{\mathcal{A}}^{(j)}$ in Eq.(72). Note that the matrix U_j are now different for each site j because its elements depend crucially on the entries of $M_{\mathcal{A}}^{(j)}$, see appendix B for details. By performing the transformation (69) we are able to turn the transfer matrix $T_g(\lambda)$ into

$$T'_g(\lambda) = \mathrm{Tr}_{\mathcal{A}}\left[\mathcal{G}_{\mathcal{A}}{}^{(ef)}\mathcal{L}_{\mathcal{A}L}(\lambda)\mathcal{L}_{\mathcal{A}L-1}(\lambda)\dots\mathcal{L}_{\mathcal{A}1}(\lambda)\right]. \tag{76}$$

Once again we end up with a transfer matrix problem with only one boundary matrix that has been discussed in details through this paper. Interesting enough, the diagonal twists entering into the Bethe ansatz equations and the transfer matrix eigenvalues are exactly the eigenvalues of the ordered product (73) of all the boundary matrices. We remark that our findings when applicable to the two-site transfer matrix $T_{1/2}^{(S_1,S_2)}(\lambda)$ of section 2.1 reproduce the eigenvalues and the Bethe ansatz equations proposed recently in ref.[18] except for the fact that the number of roots vary up to $2(S_1+S_2)$ instead of being fixed at this upper bound value.

In what follows we will present two concrete examples of the utility of quantum space transformations in the problem of transfer matrices diagonalization.

4.1 Spin-S Heisenberg model

The classical analogue of the integrable spin-S Heisenberg model [3, 4] is known to be a $2S+1$ state vertex model whose Boltzmann weights are identified with the matrix elements of the following $SU(2)$ invariant $\mathcal{L}$-operator [3, 4]

$$\mathcal{L}_{\mathcal{A}j}^{(S)}(\lambda) = \sum_{l=0}^{2S} f_l(\lambda)P_l, \tag{77}$$

where $f_l(\lambda) = (\lambda+2\eta S)\prod_{k=l+1}^{2S}\frac{\lambda-\eta k}{\lambda+\eta k}$ and P_l is the projector onto $SU(2)_l$ in the Clebsch-Gordan decomposition $SU(2)_S \otimes SU(2)_S$. This operator is conveniently represented by the expression

$$P_l = \prod_{\substack{k=0\\k\neq l}}^{2S}\frac{\vec{S}\otimes\vec{S}_j - x_k}{x_l - x_k}, \tag{78}$$

with $x_k = \frac{1}{2}\left[l(l+1) - 2S(S+1)\right]$.

As usual, the transfer matrix associated to this $SU(2)$ vertex model with an integrable boundary $\mathcal{G}_{\mathcal{A}}$ is

$$T_S(\lambda) = \mathrm{Tr}_{\mathcal{A}}\left[\mathcal{G}_{\mathcal{A}}\mathcal{L}^{(S)}_{\mathcal{A}L}(\lambda)\dots\mathcal{L}^{(S)}_{\mathcal{A}1}(\lambda)\right]. \tag{79}$$

After some computation we find that the most general boundary condition compatible with integrability can be written as

$$\mathcal{G}_{\mathcal{A}} = \begin{pmatrix} g_{S,S} & g_{S,S-1} & \cdots & g_{S,-S} \\ g_{S-1,S} & g_{S-1,S-1} & \cdots & g_{S-1,-S} \\ \vdots & \vdots & \ddots & \vdots \\ g_{-S,S} & g_{-S,S-1} & \cdots & g_{-S,-S} \end{pmatrix}, \tag{80}$$

where the coefficients $g_{m,l}$ satisfy the recurrence relation

$$g_{m,l} = \frac{\sqrt{2S(S-m)(S+1+m)}g_{S-1,S}g_{m+1,l} - \sqrt{2S(S-l)(S+1+l)}g_{S,S-1}g_{m,l+1}}{2S(l-m)g_{S,S}}, \tag{81}$$

for $l \neq m$ and $l, m = -S, \dots, S$ and

$$g_{l,l} = \frac{g_{S-1,S-1}g_{l+1,l+1}}{g_{S,S}} - \frac{\sqrt{2(2S-1)(S-l-1)(S+l+2)}g_{S-2,S}g_{l+2,l} + 2(S-l-1)g_{S-1,S}g_{l+1,l}}{\sqrt{2S(S+l+1)(S-l)}g_{S,S}}, \tag{82}$$

for $l = -S, \dots, S$.

As expected from the $SU(2)$ symmetry this boundary matrix has four free parameters represented by the elements $g_{S,S}, g_{S,S-1}, g_{S-1,S}$ and $g_{S-1,S-1}$. The standard procedure used to diagonalize transfer matrix based on higher spin representations consists in exploring its commutation with that of the mixed vertex model (10) when $S_1 = S_2 = \cdots = S_L = S$. Unfortunately, this property is no longer valid for arbitrary non-diagonal boundaries and therefore another route has to be taken in these cases. One possible way is to proceed exactly as explained in the beginning of section 4. We first find the matrix $M^{(S)}_{\mathcal{A}}$ that diagonalize the boundary matrix $\mathcal{G}_{\mathcal{A}}$ (80) and afterwards we perform the quantum space transformation (65) by using the matrices $U^{(S)}_j$ collected in the appendix B. The transformed transfer matrix $T'_S(\lambda)$

is then given by

$$T'_S(\lambda) = \mathrm{Tr}_{\mathcal{A}}\left[D^S_{\mathcal{A}}\mathcal{L}^{(S)}_{\mathcal{A}L}(\lambda)\dots\mathcal{L}^{(S)}_{\mathcal{A}1}(\lambda)\right], \tag{83}$$

where $D^S_{\mathcal{A}}$ is the diagonal matrix having the following two possible forms

$$D^S_{\mathcal{A}(+)} = \mathrm{diag}\left(g_S^{(+)}, g_S^{(-)}, \frac{\left(g_S^{(-)}\right)^2}{g_S^{(+)}}, \dots, \frac{\left(g_S^{(-)}\right)^{S-m}}{\left(g_S^{(+)}\right)^{S-m-1}}, \dots, \frac{\left(g_S^{(-)}\right)^{2S-1}}{\left(g_S^{(+)}\right)^{2S-2}}, \frac{\left(g_S^{(-)}\right)^{2S}}{\left(g_S^{(+)}\right)^{2S-1}}\right), \tag{84}$$

and

$$D^S_{\mathcal{A}(-)} = \mathrm{diag}\left(\frac{\left(g_S^{(-)}\right)^{2S}}{\left(g_S^{(+)}\right)^{2S-1}}, \frac{\left(g_S^{(-)}\right)^{2S-1}}{\left(g_S^{(+)}\right)^{2S-2}}, \dots, \frac{\left(g_S^{(-)}\right)^{S+m}}{\left(g_S^{(+)}\right)^{S+m-1}}, \dots, \frac{\left(g_S^{(-)}\right)^2}{g_S^{(+)}}, g_S^{(-)}, g_S^{(+)}\right). \tag{85}$$

The parameters $g_S^{(\pm)}$ are two particular eigenvalues that are able to parameterize the total eigenspectrum of the boundary matrix (80). We recall that these two possibilities we have at our disposal is related to a remaining $\mathcal{Z}_2$ symmetry of the Hilbert space. Now $T'_S(\lambda)$ can be diagonalized by the quantum inverse scattering method adapting the results of ref.[4] to include diagonal twists. It turns out that the final result for the eigenvalues of $T_S(\lambda)$ is

$$\Lambda^{(S)}_{n_\pm}(\lambda) = \sum_{m=-S}^{S} \frac{(g_S^{(-)})^{S\mp m}}{(g_S^{(+)})^{S\mp m-1}}\left[t_m(\lambda)\right]^L \prod_{j=1}^{n_\pm} q_m(\lambda - \lambda_j^{(\pm)} + \frac{\eta}{2}), \tag{86}$$

where the functions $t_m(\lambda)$ and $q_m(\lambda)$ are given by

$$t_m(\lambda) = (\lambda + 2\eta S)\prod_{k=m+1}^{S}\frac{\lambda + \eta k - \eta S}{\lambda + \eta k + \eta S}, \tag{87}$$

and

$$q_m(\lambda) = \frac{(\lambda + \eta S + \frac{\eta}{2})(\lambda - \eta S - \frac{\eta}{2})}{(\lambda + \eta m + \frac{\eta}{2})(\lambda + \eta m - \frac{\eta}{2})}, \tag{88}$$

while the corresponding Bethe ansatz equation for the rapidities $\lambda_j^{(\pm)}$ are

$$\left[\frac{\lambda_i^{(\pm)} + \eta S_j}{\lambda_i^{(\pm)} - \eta S_j}\right]^L = \frac{g_S^{(\mp)}}{g_S^{(\pm)}}\prod_{\substack{l=1\\ l\neq i}}^{n_\pm}\frac{\lambda_i^{(\pm)} - \lambda_l^{(\pm)} + \eta}{\lambda_i^{(\pm)} - \lambda_l^{(\pm)} - \eta}. \tag{89}$$

4.2 Singular boundary matrix

In the most part of this paper we have implicitly supposed that the boundary matrices $\mathcal{G}_{\mathcal{A}}$ are invertible. Here we would like to explore the interesting situation in which these matrices become singular. For sake of simplicity we shall discuss this problem for the transfer matrix $T_{1/2}^{(\frac{1}{2},\ldots,\frac{1}{2})}(\lambda)$ (10) that are built up in terms of the simplest $\mathcal{L}$-operators $\mathcal{L}_{\mathcal{A}j}^{(\frac{1}{2},\frac{1}{2})}(\lambda)$. A possible representation of the corresponding singular boundary matrix is

$$\mathcal{G}_{\mathcal{A}}^{(sg)} = \begin{pmatrix} g_{11} & \frac{g_{11}g_{22}}{g_{21}} \\ g_{21} & g_{22} \end{pmatrix}, \tag{90}$$

where we are assuming that $g_{21} \neq 0$.

For general values of g_{ij} this matrix is still diagonalizable and it is similar to a diagonal matrix $\mathcal{D}^{(sg)}$, namely

$$\mathcal{D}^{(sg)} = \begin{pmatrix} 0 & 0 \\ 0 & g_{11}+g_{22} \end{pmatrix}, \tag{91}$$

while the explicit expression for the matrix $M_{\mathcal{A}}$ that diagonalize $\mathcal{G}_{\mathcal{A}}^{(sg)}$ is

$$M_{\mathcal{A}}^{(sg)} = \begin{pmatrix} -\frac{g_{22}}{g_{21}} & \frac{g_{11}}{g_{21}} \\ 1 & 1 \end{pmatrix}. \tag{92}$$

We now proceed by performing the gauge transformations (67) with the help of the matrix (92) as well as the quantum space transformation (65) where $U_j = M_{\mathcal{A}}^{(sg)}$. As a result, the transformed transfer matrix we need to diagonalize becomes

$$T'^{(\frac{1}{2},\ldots,\frac{1}{2})}_{1/2}(\lambda) = \mathrm{Tr}_{\mathcal{A}}\left[D_{\mathcal{A}}^{(sg)}\mathcal{L}_{\mathcal{A}L}^{(\frac{1}{2},\frac{1}{2})}(\lambda)\ldots\mathcal{L}_{\mathcal{A}1}^{(\frac{1}{2},\frac{1}{2})}(\lambda)\right]. \tag{93}$$

Clearly, this transfer matrix is defective since $\mathcal{D}^{(sg)}$ contains one diagonal vanishing element and therefore it is proportional to a single diagonal monodromy matrix element. Thanks to this property, the non-null eigenstates $|\psi'\rangle^{(n)}$ of this operator can be found by direct inspection and they are given by

$$|\psi'\rangle^{(n)} = \prod_{i=1}^{n}\otimes\begin{pmatrix}1\\0\end{pmatrix}_i \prod_{i=n+1}^{L}\otimes\begin{pmatrix}0\\1\end{pmatrix}_i, \tag{94}$$

for $n = 0, 1, \ldots, L$. The corresponding eigenvalues $\Lambda^{(n)}(\lambda)$ have the following form

$$\Lambda^{(n)}(\lambda) = (g_{11} + g_{22})[\lambda]^n[\lambda + \eta]^{(L-n)}. \tag{95}$$

Considering the result (94) and Eq.(70) with $U_j = M_{\mathcal{A}}^{(sg)}$ we conclude that the original eigenstates of $T_{1/2}^{(\frac{1}{2},\ldots,\frac{1}{2})}(\lambda)$ are

$$|\psi\rangle^{(n)} = \prod_{i=1}^{n} \otimes \begin{pmatrix} -\frac{g_{22}}{g_{21}} \\ 1 \end{pmatrix}_i \prod_{i=n+1}^{L} \otimes \begin{pmatrix} \frac{g_{11}}{g_{21}} \\ 1 \end{pmatrix}_i . \tag{96}$$

Finally, we remark that the results (95,96) have been previously conjectured in ref.[11] on basis of exact diagonalization of $T_{1/2}^{(\frac{1}{2},\ldots,\frac{1}{2})}(\lambda)$ for some values of L. This discussion not only clarifies the origin of such results but also emphasizes the importance of the quantum space transformations in the general framework of transfer matrices diagonalization.

5 Conclusions

In this paper we have investigated integrable non-diagonal toroidal boundary conditions for mixed vertex models and related spin chains based on the isotropic $SU(N)$ R-matrix. In particular, we have applied the quantum inverse scattering method to obtain the transfer matrix eigenvalues and the corresponding Bethe ansatz equations for systems with quantum spaces invariant by the high spin representation of $SU(2)$, the discrete $D^+(k)$ representation of $SU(1,1)$ and the conjugate representation of $SU(N)$.

We have introduced the notion of quantum space transformations that together with the standard Baxter's gauge transformations [14] allowed us to diagonalize the transfer matrix of general families of vertex models that mix both boundary and $\mathcal{L}$-operator representations. We have used this approach to solve the Heisenberg model with arbitrary spin and non-diagonal twists as well as the case of singular boundary matrices. We strong believe that this method is easily applied to other isotropic vertex models such as those invariant by the $O(N)$ and $Sp(2N)$ Lie algebras [25]. We also hope that this property could be generalized to accommodate the

solution of trigonometric vertex models and therefore improving our understanding of the integrability.

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Appendix A: Completeness for $SU(2)$ mixed vertex models

In this appendix we consider the completeness of the Bethe ansatz solution of the $SU(2)$ invariant vertex models discussed in section 2.1 for $L = 2$. More precisely, we are going to verify that all the $(2S_1+1)(2S_2+1)$ eigenvalues of the transfer matrix $T_{1/2}^{(S_1,S_2)}(\lambda)$ (10) can be obtained from the Eqs.(27,28) for some values of S_1 and S_2:

- $T_{1/2}^{(\frac{1}{2},1)}(\lambda)$

We can search for solutions of the Bethe ansatz equations (28) beginning either with $|0\rangle^{(+)}$ or $|0\rangle^{(-)}$ eigenvectors. By choosing the state $|0\rangle^{(+)}$ the zero-particle eigenvalue can directly be read from (27),

$$\Lambda_0^{(\frac{1}{2},1)}(\lambda) = g_{1/2}^{(+)}(\lambda+\eta)(\lambda+\frac{3\eta}{2}) + g_{1/2}^{(-)}(\lambda)(\lambda-\frac{\eta}{2}). \tag{A.1}$$

For the one-particle state $\widetilde{B}(\lambda_1^{(+)})\,|0\rangle^{(+)}$ we find two possible Bethe ansatz roots given by

$$\lambda_{1\pm}^{(+)} = \frac{3\eta\left(g_{1/2}^{(-)}+g_{1/2}^{(+)}\right) \pm \eta\sqrt{(g_{1/2}^{(-)}+g_{1/2}^{(+)})^2+32g_{1/2}^{(-)}g_{1/2}^{(+)}}}{4\left(g_{1/2}^{(-)}-g_{1/2}^{(+)}\right)}, \tag{A.2}$$

and the respective eigenvalues are

$$\Lambda_{1\pm}^{(\frac{1}{2},1)}(\lambda) = g_{1/2}^{(+)}\left[\lambda^2+\frac{\eta}{2}\lambda+\frac{\eta^2}{4}\right] + g_{1/2}^{(-)}\left[\lambda^2+\frac{\eta}{2}\lambda-\frac{\eta^2}{4}\right] \pm \sqrt{(g_{1/2}^{(-)}+g_{1/2}^{(+)})^2+32g_{1/2}^{(+)}g_{1/2}^{(-)}}. \tag{A.3}$$

The next step is to consider the two-particle state $\widetilde{B}(\lambda_1^{(+)})\widetilde{B}(\lambda_2^{(+)})\left|0\right\rangle^{(+)}$ and again two possibilities are obtained and they are

$$\lambda_{1\pm}^{(+)} = \frac{5\eta\left(g_{1/2}^{(-)}+g_{1/2}^{(+)}\right)}{8\left(g_{1/2}^{(-)}-g_{1/2}^{(+)}\right)} \mp \frac{\eta\sqrt{(g_{1/2}^{(-)}+g_{1/2}^{(+)})^2+32g_{1/2}^{(-)}g_{1/2}^{(+)}}}{8\left(g_{1/2}^{(-)}-g_{1/2}^{(+)}\right)}$$
$$+\frac{\eta\sqrt{2}\sqrt{5({g_{1/2}^{(-)}}^2-{g_{1/2}^{(+)}}^2)-38g_{1/2}^{(-)}g_{1/2}^{(+)}\pm 3\left(g_{1/2}^{(-)}+g_{1/2}^{(+)}\right)\sqrt{(g_{1/2}^{(-)}+g_{1/2}^{(+)})^2+32g_{1/2}^{(-)}g_{1/2}^{(+)}}}}{8\left(g_{1/2}^{(-)}-g_{1/2}^{(+)}\right)}, \quad (A.4)$$

and

$$\lambda_{2\pm}^{(+)} = \frac{5\eta\left(g_{1/2}^{(-)}+g_{1/2}^{(+)}\right)}{8\left(g_{1/2}^{(-)}-g_{1/2}^{(+)}\right)} \mp \frac{\eta\sqrt{(g_{1/2}^{(-)}+g_{1/2}^{(+)})^2+32g_{1/2}^{(-)}g_{1/2}^{(+)}}}{8\left(g_{1/2}^{(-)}-g_{1/2}^{(+)}\right)}$$
$$-\frac{\eta\sqrt{2}\sqrt{5({g_{1/2}^{(-)}}^2-{g_{1/2}^{(+)}}^2)-38g_{1/2}^{(-)}g_{1/2}^{(+)}\pm 3\left(g_{1/2}^{(-)}+g_{1/2}^{(+)}\right)\sqrt{(g_{1/2}^{(-)}+g_{1/2}^{(+)})^2+32g_{1/2}^{(-)}g_{1/2}^{(+)}}}}{8\left(g_{1/2}^{(-)}-g_{1/2}^{(+)}\right)}. \quad (A.5)$$

The corresponding eigenvalues are then given by

$$\Lambda_{2\pm}^{(\frac{1}{2},1)}(\lambda) = g_{1/2}^{(-)}\left[\lambda^2+\frac{\eta}{2}\lambda+\frac{\eta^2}{4}\right]+g_{1/2}^{(+)}\left[\lambda^2+\frac{\eta}{2}\lambda-\frac{\eta^2}{4}\right]\pm\sqrt{(g_{1/2}^{(-)}+g_{1/2}^{(+)})^2+32g_{1/2}^{(+)}g_{1/2}^{(-)}}. \quad (A.6)$$

The last sector consists of three Bethe ansatz roots and an analytical expression for them has been prevented within our numerical resources. We have however performed an extensive numerical work confirming that the associated eigenvalue is

$$\Lambda_3^{(\frac{1}{2},1)}(\lambda) = g_{1/2}^{(-)}(\lambda+\eta)(\lambda+\frac{3\eta}{2})+g_{1/2}^{(+)}(\lambda)(\lambda-\frac{\eta}{2}). \quad (A.7)$$

- $T_{1/2}^{(1,1)}(\lambda)$

As before, starting with the state $\left|0\right\rangle^{(+)}$ it is not difficult to see that

$$\Lambda_0^{(1,1)}(\lambda) = g_{1/2}^{(+)}[\lambda+\frac{3\eta}{2}]^2+g_{1/2}^{(-)}[\lambda-\frac{\eta}{2}]^2. \quad (A.8)$$

The next state is the one-particle sector giving us two possibilities

$$\lambda_{1\pm}^{(+)} = -\eta\left(\frac{\sqrt{g_{1/2}^{(+)}}\pm\sqrt{g_{1/2}^{(-)}}}{\sqrt{g_{1/2}^{(+)}}\mp\sqrt{g_{1/2}^{(-)}}}\right), \quad (A.9)$$

and the corresponding eigenvalues are

$$\Lambda_{1\pm}^{(1,1)}(\lambda) = g_{1/2}^{(+)}(\lambda + \frac{3\eta}{2})(\lambda + \frac{\eta}{2}) + g_{1/2}^{(-)}(\lambda - \frac{\eta}{2})(\lambda + \frac{\eta}{2}) \pm \sqrt{g_{1/2}^{(+)} g_{1/2}^{(-)}}. \tag{A.10}$$

On the other hand the two-particle state provides us three types of Bethe ansatz roots. The first one is

$$\lambda_1^{(+)} = \eta \frac{(g_{1/2}^{(-)} + g_{1/2}^{(+)})}{(g_{1/2}^{(-)} - g_{1/2}^{(+)})} \quad \text{and} \quad \lambda_2^{(+)} = 0, \tag{A.11}$$

and the respective eigenvalue is

$$\Lambda_2^{(1,1)}(\lambda) = \left(g_{1/2}^{(+)} + g_{1/2}^{(-)}\right)(\lambda + \frac{3\eta}{2})(\lambda - \frac{\eta}{2}). \tag{A.12}$$

The other remaining solutions are given by

$$\lambda_{1\pm}^{(+)} = \frac{3\eta(g_{1/2}^{(-)} + g_{1/2}^{(+)})}{4(g_{1/2}^{(-)} - g_{1/2}^{(+)})} \mp \eta \frac{\sqrt{(g_{1/2}^{(+)} + g_{1/2}^{(-)})^2 + 32 g_{1/2}^{(+)} g_{1/2}^{(-)}}}{4(g_{1/2}^{(-)} - g_{1/2}^{(+)})}$$
$$+\eta \frac{\sqrt{(g_{1/2}^{(+)} + g_{1/2}^{(-)})^2 - 16 g_{1/2}^{(+)} g_{1/2}^{(-)} \pm ({g_{1/2}^{(-)}}^2 - {g_{1/2}^{(+)}}^2)\sqrt{(g_{1/2}^{(+)} + g_{1/2}^{(-)})^2 + 32 g_{1/2}^{(+)} g_{1/2}^{(-)}}}}{2\sqrt{2}(g_{1/2}^{(-)} - g_{1/2}^{(+)})}, \tag{A.13}$$

and

$$\lambda_{2\pm}^{(+)} = \frac{3\eta(g_{1/2}^{(-)} + g_{1/2}^{(+)})}{4(g_{1/2}^{(-)} - g_{1/2}^{(+)})} \mp \eta \frac{\sqrt{(g_{1/2}^{(+)} + g_{1/2}^{(-)})^2 + 32 g_{1/2}^{(+)} g_{1/2}^{(-)}}}{4(g_{1/2}^{(-)} - g_{1/2}^{(+)})}$$
$$-\eta \frac{\sqrt{(g_{1/2}^{(+)} + g_{1/2}^{(-)})^2 - 16 g_{1/2}^{(+)} g_{1/2}^{(-)} \pm ({g_{1/2}^{(-)}}^2 - {g_{1/2}^{(+)}}^2)\sqrt{(g_{1/2}^{(+)} + g_{1/2}^{(-)})^2 + 32 g_{1/2}^{(+)} g_{1/2}^{(-)}}}}{2\sqrt{2}(g_{1/2}^{(-)} - g_{1/2}^{(+)})}, \tag{A.14}$$

while the associated eigenvalues are

$$\Lambda_{2\pm}^{(1,1)}(\lambda) = \left(g_{1/2}^{(+)} + g_{1/2}^{(-)}\right)\left[(\lambda + \frac{3\eta}{2})(\lambda - \frac{\eta}{2}) + \frac{\eta}{2}\right] \pm \frac{1}{2}\sqrt{(g_{1/2}^{(+)} + g_{1/2}^{(-)})^2 + 32 g_{1/2}^{(+)} g_{1/2}^{(-)}}. \tag{A.15}$$

The remaining sectors have been investigated numerically and we verified that the three and the four particle states produce the following eigenvalues

$$\Lambda_{3\pm}^{(1,1)}(\lambda) = g_{1/2}^{(-)}(\lambda + \frac{3\eta}{2})(\lambda + \frac{\eta}{2}) + g_{1/2}^{(+)}(\lambda - \frac{\eta}{2})(\lambda + \frac{\eta}{2}) \pm \sqrt{g_{1/2}^{(+)} g_{1/2}^{(-)}}, \tag{A.16}$$

and

$$\Lambda_4^{(1,1)}(\lambda) = g_{1/2}^{(-)}[\lambda + \frac{3\eta}{2}]^2 + g_{1/2}^{(+)}[\lambda - \frac{\eta}{2}]^2. \tag{A.17}$$

Finally, we remark that as byproduct of this analysis we have also checked the completeness of the eigenspectrum of the transfer matrix $T_1^{(1,1)}(\lambda)$ of section 4.1 since its Bethe ansatz equation are the same as that discussed above.

Appendix B: Quantum space transformation

In this appendix we will present the explicit expressions for the quantum space transformations U_j that are able to undo the gauge transformations on the auxiliary space. The matrix elements of U_j are directly related to entries of the gauge matrix $M_{\mathcal{A}}^{(j)}$ since they have been required to satisfy the following identity

$$\mathcal{L}_{\mathcal{A}j}(\lambda) = U_j^{-1}[M_{\mathcal{A}}^{(j)}]^{-1}\mathcal{L}_{\mathcal{A}j}(\lambda)M_{\mathcal{A}}^{(j)}U_j, \tag{B.1}$$

for each $\mathcal{L}$-operator mentioned in the main text. In order to present this relationship it is convenient to write the gauge matrix as

$$M_{\mathcal{A}}^{(j)} = \sum_{\alpha,\beta=1}^{N} \bar{m}_{\alpha,\beta}^{(j)} \hat{e}_{\alpha,\beta}^{(\mathcal{A})}, \tag{B.2}$$

where we recall that N is the dimension of the auxiliary space $\mathcal{A}$. We now start to list the quantum matrices U_j for the $\mathcal{L}$-operators used in this paper

- For $\mathcal{L}_{\mathcal{A}j}^{(\frac{1}{2},S_j)}(\lambda)$:

$$U_j^{(\frac{1}{2},S_j)} = \begin{pmatrix} u_{S_j,S_j} & u_{S_j,S_j-1} & \cdots & u_{S_j,-S_j} \\ u_{S_j-1,S_j} & u_{S_j-1,S_j-1} & \cdots & u_{S_j-1,-S_j} \\ \vdots & \vdots & \ddots & \vdots \\ u_{-S_j,S_j} & u_{-S_j,S_j-1} & \cdots & u_{-S_j,-S_j} \end{pmatrix}, \tag{B.3}$$

where some of the matrix elements $u_{i,j}$ are

$$u_{S_j,S_j-k} = [\bar{m}_{1,1}^{(j)}]^{2S_j-k}[\bar{m}_{1,2}^{(j)}]^k \sqrt{\frac{(2S_j)!}{k!(2S_j-k)!}}, \tag{B.4}$$

while the remaining ones satisfy the following recurrence relation

$$
\begin{aligned}
u_{S_j-k-1,S_j-n} &= \frac{(\bar{m}^{(j)}_{1,1}\bar{m}^{(j)}_{2,2}-\bar{m}^{(j)}_{1,2}\bar{m}^{(j)}_{2,1})}{[\bar{m}^{(j)}_{1,1}]^2}\sqrt{\frac{n(2S_j-n+1)}{(k+1)(2S_j-k)}}u_{S_j-k,S_j-n+1} \\
&+\left(\frac{\bar{m}^{(j)}_{2,1}}{\bar{m}^{(j)}_{1,1}}\right)^2\sqrt{\frac{k(2S_j-k+1)}{(k+1)(2S_j-k)}}u_{S_j-k+1,S_j-n}+\frac{\bar{m}^{(j)}_{2,1}}{\bar{m}^{(j)}_{1,1}}\frac{2(S_j-k)}{\sqrt{(k+1)(2S_j-k)}}u_{S_j-k,S_j-n},
\end{aligned} \tag{B.5}
$$

and k, n are integers satisfying $k, n = 0, \ldots, 2S_j$.

- For $\mathcal{L}^{(\frac{1}{2},k_j)}_{\mathcal{A}j}(\lambda)$:

$$
U^{(\frac{1}{2},k_j)}_j = \begin{pmatrix} \bar{u}_{0,0} & \bar{u}_{0,1} & \bar{u}_{0,2} & \cdots \\ \bar{u}_{1,0} & \bar{u}_{1,1} & \bar{u}_{1,2} & \cdots \\ \bar{u}_{2,0} & \bar{u}_{2,1} & \bar{u}_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{B.6}
$$

where the first row are given by

$$
\bar{u}_{0,n} = (-1)^n\left(\frac{\bar{m}^{(j)}_{2,1}}{\bar{m}^{(j)}_{2,2}}\right)^n\sqrt{\frac{(2k_j+n-1)!}{n!(2k_j-1)!}}\bar{u}_{0,0} \tag{B.7}
$$

and the other elements are obtained recursively by the following relation

$$
\begin{aligned}
\bar{u}_{n+1,l} &= \frac{(\bar{m}^{(j)}_{1,1}\bar{m}^{(j)}_{2,2}-\bar{m}^{(j)}_{1,2}\bar{m}^{(j)}_{2,1})}{[\bar{m}^{(j)}_{2,2}]^2}\sqrt{\frac{l(2k_j+l-1)}{(n+1)(2k_j+n)}}\bar{u}_{n,l-1} \\
&-\left(\frac{\bar{m}^{(j)}_{1,2}}{\bar{m}^{(j)}_{2,2}}\right)^2\sqrt{\frac{n(2k_j+n-1)}{(n+1)(2k_j+n)}}\bar{u}_{n-1,l}+\frac{\bar{m}^{(j)}_{1,2}}{\bar{m}^{(j)}_{2,2}}\frac{2(k_j+n)}{\sqrt{(n+1)(2k_j+n)}}\bar{u}_{n,l},
\end{aligned} \tag{B.8}
$$

and n, l are integers $n, l = 0, 1, 2, \ldots$.

- For the higher spin operator $\mathcal{L}^{(S)}_{\mathcal{A}j}(\lambda)$:

$$
U^{(S)}_j = M^{(S)}_{\mathcal{A}} \tag{B.9}
$$

- For the fundamental $SU(N)$ operator $\mathcal{L}^{(1)}_{\mathcal{A}j}(\lambda)$:

$$
U^{(1)}_j = M^{(j)}_{\mathcal{A}} \tag{B.10}
$$

- For the conjugated $SU(N)$ operator $\mathcal{L}^{(2)}_{\mathcal{A}j}(\lambda)$:

$$U^{(2)}_j = \begin{pmatrix} \bar{c}^{(j)}_{N,N} & \bar{c}^{(j)}_{N,N-1} & \cdots & \bar{c}^{(j)}_{N,1} \\ \bar{c}^{(j)}_{N-1,N} & \bar{c}^{(j)}_{N-1,N-1} & \cdots & \bar{c}^{(j)}_{N-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{c}^{(j)}_{1,N} & \bar{c}^{(j)}_{1,N-1} & \cdots & \bar{c}^{(j)}_{1,1} \end{pmatrix}, \tag{B.11}$$

where $\bar{c}^{(j)}_{\alpha,\beta}$ are the following cofactors

$$\bar{c}^{(j)}_{\alpha,\beta} = (-1)^{\alpha+\beta} \begin{vmatrix} \bar{m}^{(j)}_{1,1} & \dots & \bar{m}^{(j)}_{1,\beta-1} & \bar{m}^{(j)}_{1,\beta+1} & \dots & \bar{m}^{(j)}_{1,N} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ \bar{m}^{(j)}_{\alpha-1,1} & \dots & \bar{m}^{(j)}_{\alpha-1,\beta-1} & \bar{m}^{(j)}_{\alpha-1,\beta+1} & \dots & \bar{m}^{(j)}_{\alpha-1,N} \\ \bar{m}^{(j)}_{\alpha+1,1} & \dots & \bar{m}^{(j)}_{\alpha+1,\beta-1} & \bar{m}^{(j)}_{\alpha+1,\beta+1} & \dots & \bar{m}^{(j)}_{\alpha+1,N} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ \bar{m}^{(j)}_{N,1} & \dots & \bar{m}^{(j)}_{N,\beta-1} & \bar{m}^{(j)}_{N,\beta+1} & \dots & \bar{m}^{(j)}_{N,N} \end{vmatrix}. \tag{B.12}$$

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